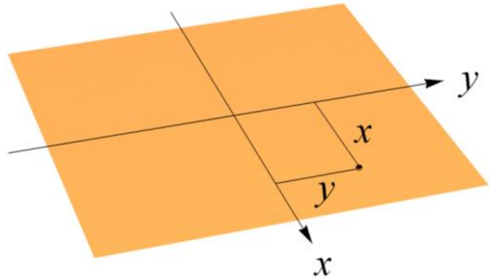
Cartesian Coordinates



An oldie but a goodie, yet not always the best choice!

Area of a circle in Cartesian coordinates

$$\int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx\,dy \overset{pain}{\Longrightarrow} \pi R^2$$

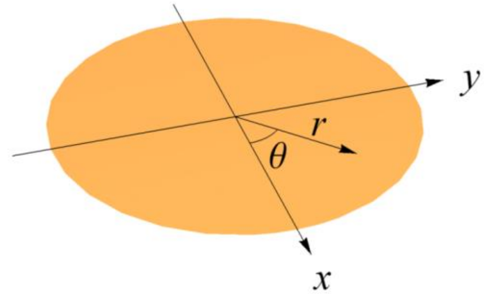
Area of a circle in Polar coordinates

$$\int_{0}^{2\pi}\int_{0}^{R} r\,dr\,d\theta \overset{easy}{\Longrightarrow} \pi R^2$$

Area Element

$$dV = dx\,dy$$

Polar Coordinates



Polar to Cartesian

$$\begin{bmatrix} x = r\,\text{Cos}[\theta] \\ y = r\,\text{Sin}[\theta] \end{bmatrix}$$

$$\begin{bmatrix} r = \sqrt{x^2+y^2} \\ \theta = \text{ArcTan}\left[\frac{y}{x}\right] \end{bmatrix}$$

Cartesian to Polar

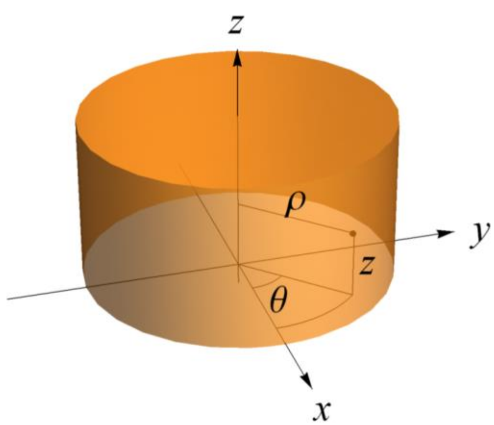
Unit Vectors

$$\begin{bmatrix} \hat{x} = \text{Cos}[\theta]\,\hat{\rho} - \text{Sin}[\theta]\,\hat{\theta} \\ \hat{y} = \text{Sin}[\theta]\,\hat{\rho} + \text{Cos}[\theta]\,\hat{\theta} \end{bmatrix} \begin{bmatrix} \hat{r} = \text{Cos}[\theta]\,\hat{x} + \text{Sin}[\theta]\,\hat{y} \\ \hat{\theta} = -\text{Sin}[\theta]\,\hat{x} + \text{Cos}[\theta]\,\hat{y} \end{bmatrix}$$

Area Element

$$dV = r\,dr\,d\theta$$

Cylindrical Coordinates

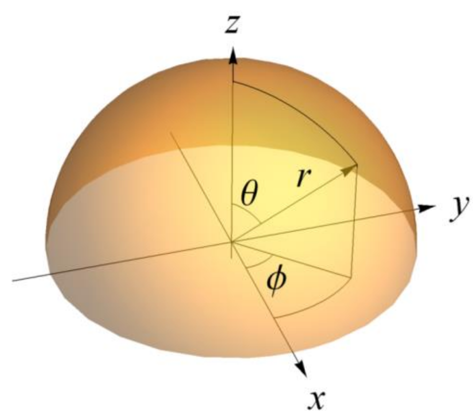


$$\begin{bmatrix} x = \rho\,\text{Cos}[\theta] \\ y = \rho\,\text{Sin}[\theta] \\ z = z \end{bmatrix} \quad \begin{bmatrix} \rho = \sqrt{x^2+y^2} \\ \theta = \text{ArcTan}\left[\frac{y}{x}\right] \\ z = z \end{bmatrix}$$

Volume Element

$$dV = r^2\,\text{Sin}[\theta]\,dr\,d\theta\,d\phi$$

Spherical Coordinates



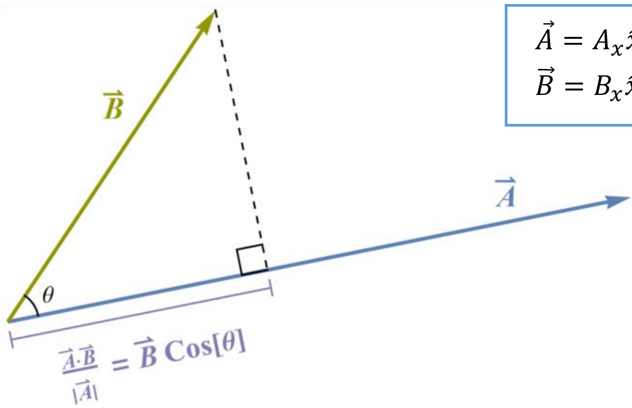
$$\begin{bmatrix} x = r\,\text{Sin}[\theta]\,\text{Cos}[\phi] \\ y = r\,\text{Sin}[\theta]\,\text{Sin}[\phi] \\ z = r\,\text{Cos}[\theta] \end{bmatrix} \begin{bmatrix} r = \sqrt{x^2+y^2+z^2} \\ \theta = \text{ArcTan}\left[\frac{(x^2+y^2)^{1/2}}{z}\right] \\ \phi = \text{ArcTan}\left[\frac{y}{x}\right] \end{bmatrix}$$

Volume Element

$$dV = r^2\,\text{Sin}[\theta]\,dr\,d\theta\,d\phi$$

Dot Product

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$
$$= |\vec{A}||\vec{B}|\text{Cos}[\theta]$$

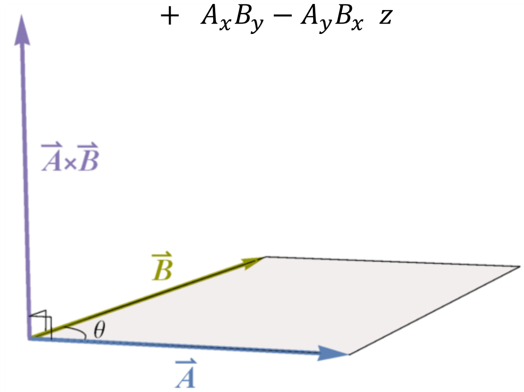


$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

- $\vec{A}\cdot\vec{B}$ is a *scalar*
- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} \longrightarrow \vec{A}\cdot\vec{A} = |\vec{A}|^2$
- For any *unit* vector $\hat{n}$, $\vec{A}\cdot\hat{n}$ represents the length of $\vec{A}$ along the direction $\hat{n}$

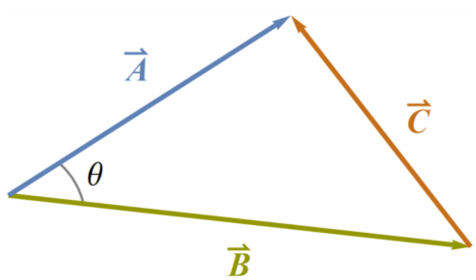
Cross Product

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{x}$$
$$+\ A_zB_x - A_xB_z\ y$$
$$+\ A_xB_y - A_yB_x\ z$$



- $\vec{A}\times\vec{B}$ is a *vector*
- $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A} \longrightarrow \vec{A}\times\vec{A} = \vec{0}$
- Direction given by right-hand rule and magnitude $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\text{Sin}[\theta]$ equal to parallelogram area

Law of Cosines



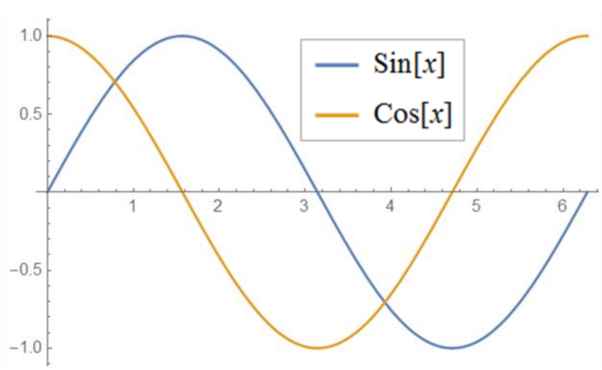
- Triangle defined by vectors $\vec{A}$ and $\vec{B}$
- Thirds leg given by $\vec{C} = \vec{A} - \vec{B}$

$$|\vec{\boldsymbol{C}}|^2 = |\vec{\boldsymbol{A}}|^2 + |\vec{\boldsymbol{B}}|^2 - 2|\vec{\boldsymbol{A}}||\vec{\boldsymbol{B}}|\textbf{Cos}[\boldsymbol{\theta}]$$

Proof

$$\vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$$
$$= |\vec{A}|^2 + |\vec{B}|^2 - 2\,\vec{A}\cdot\vec{B}$$

Trig Functions



$$\text{Cos}[0] = 1 \qquad \text{Sin}[0] = 0$$
$$\text{Cos}\left[\frac{\pi}{6}\right] = \frac{\sqrt{3}}{2} \qquad \text{Sin}\left[\frac{\pi}{6}\right] = \frac{1}{2}$$
$$\text{Cos}\left[\frac{\pi}{4}\right] = \frac{\sqrt{2}}{2} \qquad \text{Sin}\left[\frac{\pi}{4}\right] = \frac{\sqrt{2}}{2}$$
$$\text{Cos}\left[\frac{\pi}{3}\right] = \frac{1}{2} \qquad \text{Sin}\left[\frac{\pi}{3}\right] = \frac{\sqrt{3}}{2}$$
$$\text{Cos}\left[\frac{\pi}{2}\right] = 0 \qquad \text{Sin}\left[\frac{\pi}{2}\right] = 1$$
$$\text{Cos}[\pi - x] = -\text{Cos}[x] \qquad \text{Sin}[\pi - x] = \text{Sin}[x]$$

$$\text{Sin}[x] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{j=0}^{\infty}(-1)^j\frac{x^{2j+1}}{(2j+1)!} = \frac{e^{ix} - e^{-ix}}{2}$$

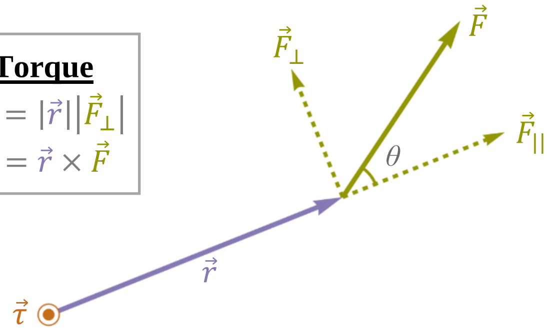
$$\text{Cos}[x] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{j=0}^{\infty}(-1)^j\frac{x^{2j}}{(2j)!} = \frac{e^{ix} + e^{-ix}}{2}$$

$$\text{Sin}[x+y] = \text{Sin}[x]\text{Cos}[y] + \text{Cos}[x]\text{Sin}[y]$$
$$\text{Cos}[x+y] = \text{Cos}[x]\text{Cos}[y] - \text{Sin}[x]\text{Sin}[y]$$
$$\text{Sin}[2x] = 2\text{Sin}[x]\text{Cos}[x]$$
$$\text{Cos}[2x] = \text{Cos}[x]^2 - \text{Sin}[x]^2$$
$$= 2\text{Cos}[x]^2 - 1 = 1 - 2\text{Sin}[x]^2$$

Torque

$|\vec{\tau}| = |\vec{r}||\vec{F}_\perp|$

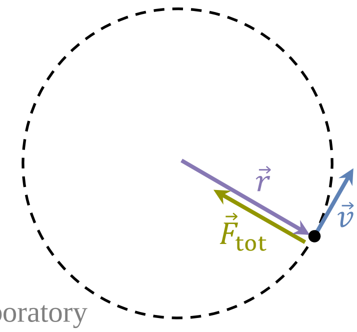
$\vec{\tau} = \vec{r} \times \vec{F}$



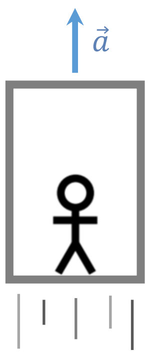
- Bigger wrench yields more torque
- Force pointing at base point has $\vec{\tau} = \vec{0}$

Circular Motion

$\vec{F}_{\text{tot}} = -\frac{mv^2}{r}\hat{r}$



- Ex: Geostationary orbit, laboratory centrifuge, playground carousel
- Tighter circles require larger $\vec{F}_{\text{tot}}$



Linearly Accelerating Frame

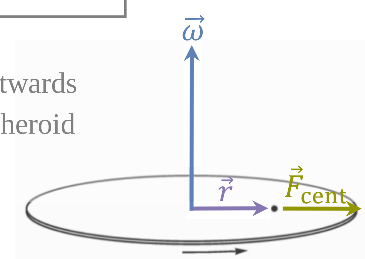
Fictitious force $\vec{F}_{\text{linear}} = -m\vec{a}$

- Bus starts moving → Pushed into seat
- Elevator moves up → You feel heavier

Rotating Reference Frame

Fictitious force $\vec{F}_{\text{cent}} = m\omega^2\vec{r}$

- Turn in a car → Pushed outwards
- Earth rotating → Oblate spheroid

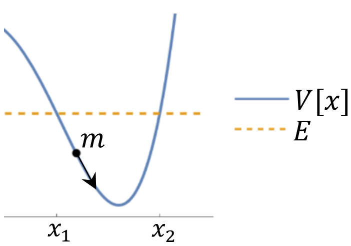


Newton's Laws

1st: $\vec{a} = \vec{0}$ implies $\vec{v} =$ constant

2nd: $\sum \vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$ (constant m)

3rd: Equal and opposite forces



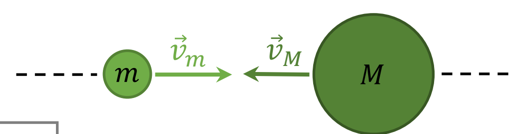
Energy

$E = \frac{1}{2}mv^2 + V[x]$

$F[x] = -\frac{dV}{dx}$

Mass in potential $V[x]$

- m oscillates between x_1 and x_2
- $v = 0$ when $E = V[x_1] = V[x_2]$



Momentum

$\vec{p} = m\vec{v}$

Elastic collision with $m = M$

- Masses swap velocities

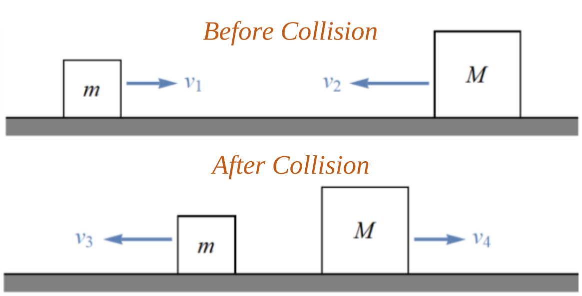
Elastic collision with $m \ll M$

- M's velocity nearly unchanged
- m's velocity increases to $v_m + 2v_M$

Elastic Collision

Masses rebound elastically → Energy conserved

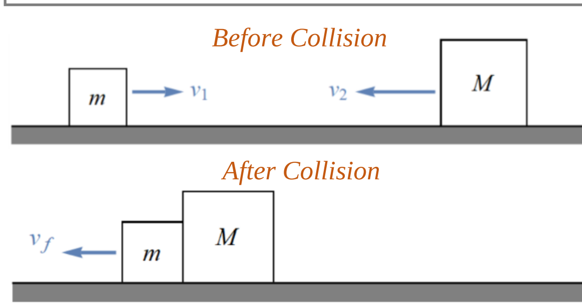
No net force → Momentum conserved



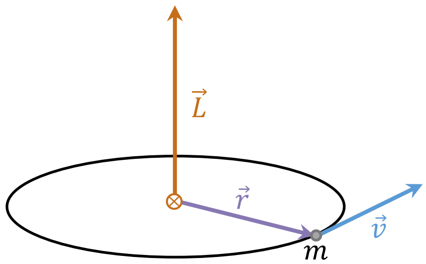
Inelastic Collision

Masses stick together → Energy *not* conserved

No net force → Momentum conserved



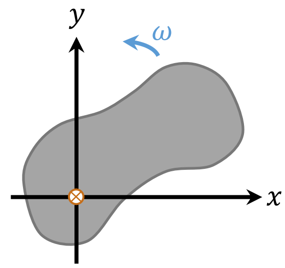
Angular Momentum and Kinetic Energy



Point mass

$\vec{L} = m\vec{r} \times \vec{v}$

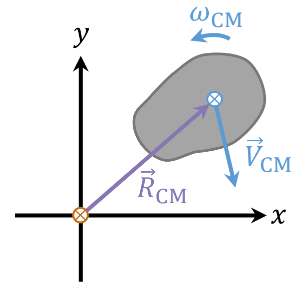
$\mathrm{KE} = \frac{1}{2}mv^2$



Extended body, pure rotation

$\vec{L} = I\vec{\omega}$

$\mathrm{KE} = \frac{1}{2}I\omega^2$



Extended body, general motion

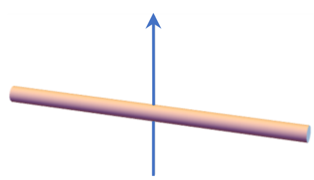
$\vec{L} = M\vec{R}_{\mathrm{CM}} \times \vec{V}_{\mathrm{CM}} + I_{\mathrm{CM}}\vec{\omega}_{\mathrm{CM}}$

$\mathrm{KE} = \frac{1}{2}MV_{\mathrm{CM}}^2 + \frac{1}{2}I_{\mathrm{CM}}\omega_{\mathrm{CM}}^2$

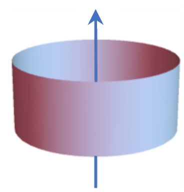
Moment of Inertia

$I = \int r^2 dm$ *(About an axis)*

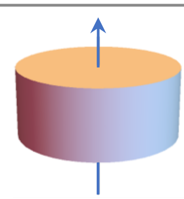
1D Rod $\frac{1}{12}ML^2$



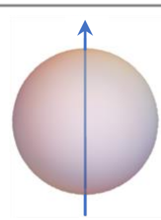
Hollow Cylinder MR^2



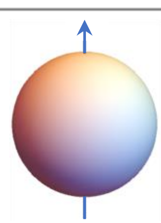
Solid Cylinder $\frac{1}{2}MR^2$



Hollow Sphere $\frac{2}{3}MR^2$



Solid Cylinder $\frac{2}{5}MR^2$



Center of Mass

$$\vec{R}_{\mathrm{CM}} = \frac{\sum_j m_j \vec{r}_j}{\sum_j m_j}, \quad \vec{V}_{\mathrm{CM}} = \frac{\sum_j m_j \vec{v}_j}{\sum_j m_j}$$

Parallel Axis Theorem

$$I = I_{\mathrm{CM}} + Md^2$$

Example (1D Rod): $I_{\mathrm{CM}} = \frac{1}{12}ML^2$

About end, $I = I_{\mathrm{CM}} + M\left(\frac{l}{2}\right)^2 = \frac{1}{3}ML^2$

Perpendicular Axis Theorem

(For flat objects on x-y plane)

$$I_z = I_x + I_y$$

Example (2D Disk): $I_z = \frac{1}{2}MR^2$ out of plane

$I_x = I_y$ by symmetry $\rightarrow I_x = I_y = \frac{1}{4}MR^2$

Rotational Dynamics

$$\sum\vec{\tau} = \frac{d\vec{L}}{dt} = I\alpha \quad \text{(pure rotation)}$$